

Quantum Limits of Measurements and Uncertainty Principle

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1. Introduction

In the theoretical considerations on quantum aspects of optical communications, one of the most important programs is to establish limits of measurements which are subjected to the laws of quantum mechanics in a rigorous and unified manner. In such a program, it is natural to expect that the uncertainty principle will play a central role. However, the recent controversy [28, 5, 21, 22], which arose in the field of gravitational-wave detection, on the validity of the standard quantum limit for monitoring of the free-mass position [4, 6] revealed a certain weakness of our understanding of the Heisenberg uncertainty principle.

Historically, Heisenberg established the uncertainty principle from his analysis of *gedanken* experiments for certain processes of successive measurements [10]. Thus his uncertainty principle is often stated [25, p. 239] in a form that a measurement of one variable from a conjugate pair disturbs the value of the conjugate variable no less than the order of $\hbar/(\text{accuracy of the measurement})$. Nevertheless, we have not established a general theory of this kind of uncertainty principle, as pointed out by several authors [3, 13].

Our uncertainty relation in current text books was first proved by Robertson [24] by a simple mathematical reasoning with use of the Schwarz inequality. However, it is often pointed out that Robertson's uncertainty principle does not mean the Heisenberg uncertainty principle. Robertson's uncertainty principle is only concerned with state preparations as in the statement that any state preparation gives an ensemble of objects in which the product of the standard deviations of conjugate variables is greater than $\hbar/2$ [3, 12].

In this paper, we shall show how the Robertson uncertainty relation gives certain intrinsic quantum limits of measurements in the most general and rigorous mathematical treatment. In Section 2, fragments from our previous work on mathematical foundations of quantum probability theory are given (see, for the detail, [15, 16, 17, 18, 19, 20]). In Section 3, some basic properties of root-mean-square error of measurement, called precision, introduced in [21] is examined and, in Section 4, a general lower bound of the product of precisions arising in joint measurements of noncommuting observables is established. This result is used to give a general proof of the uncertainty relation for the joint measurements which has been found by several authors [2, 11, 27, 1]. In Section 5, we shall give a rigorous condition for holding of the standard quantum limit (SQL) for

repeated measurements. For this purpose, we shall examine another root-mean-square error, called resolution, introduced in [21] and prove that if a measuring instruments has no larger resolution than the precision then it obeys the SQL. As shown in [21, 22, 23], we can even construct many linear models of position measurement which circumvent the above condition. In Section 6, some conclusions from the present analysis will be discussed.

2. Foundations of quantum probability

Let \mathcal{H} be a Hilbert space. Denote by $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on \mathcal{H} , by $\tau c(\mathcal{H})$ the space of trace class operators on \mathcal{H} and by $\sigma c(\mathcal{H})$ the space of Hilbert-Schmidt class operators on \mathcal{H} . A positive operator in $\tau c(\mathcal{H})$ with the unit trace is called a *density operator* and $\mathcal{S}(\mathcal{H})$ stands for the space of density operators on \mathcal{H} . Denote by $\mathcal{B}(\mathbf{R}^d)$ the Borel σ -field of the Euclidean space \mathbf{R}^d . A map $X : \mathcal{B}(\mathbf{R}^d) \rightarrow \mathcal{L}(\mathcal{H})$ is called a *probability-operator-valued measure* (POM) if it satisfies the following conditions (P1)–(P2):

(P1) For any sequence $\langle \Delta_i \mid i = 1, 2, \dots \rangle$ of disjoint sets in $\mathcal{B}(\mathbf{R}^d)$,

$$X\left(\bigcup_{i=1}^{\infty} \Delta_i\right) = \sum_{i=1}^{\infty} X(\Delta_i),$$

where the sum is convergent in the weak operator topology.

(P2) $X(\mathbf{R}) = 1$.

A linear transformation $T : \tau c(\mathcal{H}) \rightarrow \tau c(\mathcal{H})$ is called a *positive map* if $T(\rho) \geq 0$ for all $\rho \in \mathcal{S}(\mathcal{H})$. We shall denote the space of all positive maps on $\tau c(\mathcal{H})$ by $P(\tau c(\mathcal{H}))$. A map $\mathbf{X} : \mathcal{B}(\mathbf{R}^d) \rightarrow P(\tau c(\mathcal{H}))$ is called an *operation-valued measure* if it satisfies the following conditions (O1)–(O2):

(O1) For any sequence $\langle \Delta_i \mid i = 1, 2, \dots \rangle$ of disjoint sets in $\mathcal{B}(\mathbf{R}^d)$,

$$\mathbf{X}\left(\bigcup_{i=1}^{\infty} \Delta_i\right) = \sum_{i=1}^{\infty} \mathbf{X}(\Delta_i),$$

where the sum is convergent in the strong operator topology of $P(\tau c(\mathcal{H}))$.

(O2) For any $\rho \in \tau c(\mathcal{H})$,

$$\text{Tr}[\mathbf{X}(\mathbf{R}^d)\rho] = \text{Tr}[\rho].$$

An operation-valued measure $\mathbf{X} : \mathcal{B}(\mathbf{R}^d) \rightarrow P(\tau c(\mathcal{H}))$ is called a *completely positive operation-valued measure* (CPOM) if it satisfies the following condition (O3):

(O3) For any $\Delta \in \mathcal{B}(\mathbf{R}^d)$, $\mathbf{X}(\Delta)$ is a completely positive map on $\tau c(\mathcal{H})$, i.e.,

$$\sum_{i,j=1}^n \langle \xi_i | \mathbf{X}(\Delta) (|\eta_i\rangle\langle\eta_j|) | \xi_j \rangle \geq 0,$$

for all $\Delta \in \mathcal{B}(\mathbf{R}^d)$ and for all finite sequences ξ_1, \dots, ξ_n and η_1, \dots, η_n in \mathcal{H} .

The *transpose* ${}^tT : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ of $T \in \mathcal{P}(\tau c(\mathcal{H}))$ is defined by the relation

$$\mathrm{Tr}[{}^tT(a)\rho] = \mathrm{Tr}[aT(\rho)],$$

for all $a \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$. In this case, tT is also positive in the sense that $T(a) \geq 0$ for any $a \geq 0$ in $\mathcal{L}(\mathcal{H})$. For any operation-valued measure \mathbf{X} , the relation

$$\hat{\mathbf{X}}(\Delta) = {}^t\mathbf{X}(\Delta)1 \quad (\Delta \in \mathcal{B}(\mathbf{R}^d)),$$

determines a POM $\hat{\mathbf{X}}$, called the POM *associated* with \mathbf{X} . Conversely, any POM X has at least one CPOM \mathbf{X} such that $X = \hat{\mathbf{X}}$ [16, Proposition 4.1]. POM's are called “measurements” in [12] and operation-valued measures are called “instruments” in [8, 7]. Our terminology is intended to be more neutral in meanings in the physical context.

Suppose that a Hilbert space \mathcal{H} is the state space of a quantum system \mathbf{S} . A *state* of \mathbf{S} is a density operator on \mathcal{H} and an *observable* of \mathbf{S} is a POM $A : \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{L}(\mathcal{H})$ such that $A(\Delta)$ is a projection for all $\Delta \in \mathcal{B}(\mathbf{R})$. A state of the form $|\psi\rangle\langle\psi|$ for a unit vector $\psi \in \mathcal{H}$ is called a *pure state* and ψ is called a *vector state* of \mathbf{S} . A finite set $\{A_1, \dots, A_n\}$ of observables is called *compatible* if $[A_i(\Delta_1), A_j(\Delta_2)] = 0$ for all $i, j = 1, \dots, n$ and $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$. The *joint probability distribution* of a compatible set $\{A_1, \dots, A_n\}$ of observables in a state ρ , denoted by $\mathrm{Pr}[A_1 \in \Delta_1, \dots, A_n \in \Delta_n | \rho]$, $(\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R}))$, is given by the following *Born statistical formula*:

$$\mathrm{Pr}[A_1 \in \Delta_1, \dots, A_n \in \Delta_n | \rho] = \mathrm{Tr}[A_1(\Delta_1) \cdots A_n(\Delta_n)\rho].$$

The symbol ρ in the left-hand-side denotes the state for which the probability distribution is determined. For pure states $\rho = |\psi\rangle\langle\psi|$, the symbol ψ will be sometimes used instead of ρ in this and similar expressions, and then we have

$$\mathrm{Pr}[A_1 \in \Delta_1, \dots, A_n \in \Delta_n | \psi] = \langle\psi|A_1(\Delta_1) \cdots A_n(\Delta_n)|\psi\rangle.$$

By spectral theory, the relation

$$\hat{A} = \int_{\mathbf{R}} aA(da).$$

sets up a one-to-one correspondence between observables A and self-adjoint operators \hat{A} .

In this paper, by a *measurement* we shall mean generally an experiment described as follows. Let \mathbf{P} be a quantum system, called a *probe system*, described by a Hilbert space \mathcal{K} . The system \mathbf{P} is coupled to the system \mathbf{S} during a finite time interval from time t to $t + \tilde{\tau}$. Denote by \hat{U} the unitary operator on $\mathcal{H} \otimes \mathcal{K}$ corresponding to the time evolution of the system $\mathbf{S} + \mathbf{P}$ from time t to $t + \tilde{\tau}$. The time t is called the *time of measurement* and the time $t + \tilde{\tau}$ is called the *time just after measurement*. At the time just after the measurement, the systems \mathbf{S} and \mathbf{P} are separated and in order to obtain the outcome of this experiment a compatible sequence $\langle M_1, \dots, M_n \rangle$ of observables of the system \mathbf{P} are measured by an ideal manner. The observables M_1, \dots, M_n are called the *meter observables*. In order to assure the reproducibility of this experiment, the probe system \mathbf{P} is always prepared in a fixed state σ at the time of measurement. Thus the physical process of a given measurement is characterized by a 4-tuple $\mathcal{X} = [\mathcal{K}, \sigma, \hat{U}, \langle \hat{M}_1, \dots, \hat{M}_n \rangle]$, called a *measurement scheme*, consisting of a Hilbert space \mathcal{K} , a density operator σ on \mathcal{K} , a unitary operator \hat{U} on $\mathcal{H} \otimes \mathcal{K}$ and a compatible sequence $\langle \hat{M}_1, \dots, \hat{M}_n \rangle$ of self-adjoint

operators on \mathcal{K} . Every measurement scheme $\mathcal{X} = [\mathcal{K}, \sigma, \hat{U}, \langle \hat{M}_1, \dots, \hat{M}_n \rangle]$ determines a unique CPOM $\mathbf{X} : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{P}(\tau\mathcal{C}(\mathcal{H}))$, called the CPOM of \mathcal{X} , by the following relation

$$\mathbf{X}(\Delta_1 \times \dots \times \Delta_n)\rho = \text{Tr}_{\mathcal{K}} \left[(1 \otimes M_1(\Delta_1) \cdots M_n(\Delta_n)) \hat{U}(\rho \otimes \sigma) \hat{U}^\dagger \right], \quad (2.1)$$

for all $\rho \in \tau\mathcal{C}(\mathcal{H})$ and $\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R})$, where $\text{Tr}_{\mathcal{K}}$ stands for the partial trace operation of \mathcal{K} . Then the CPOM \mathbf{X} satisfies the following *Davies-Lewis postulates* (DL1)–(DL2) (cf. [8]):

(DL1) *Measurement probability*: If the state of the system \mathbf{S} at the time of measurement is ρ , then the probability distribution $\text{Pr}[X \in \Delta | \rho]$, ($\Delta \in \mathcal{B}(\mathbf{R}^n)$), of the outcome variable X of the measurement is

$$\text{Pr}[X \in \Delta | \rho] = \text{Tr}[\mathbf{X}(\Delta)\rho].$$

(DL2) *State reduction*: If the state of the system \mathbf{S} at the time of measurement is ρ , then the measurement changes the state so that the state ρ_Δ , at the time just after measurement, of the subensemble \mathbf{S}_Δ of the systems selected by the condition $X \in \Delta$ is given by

$$\rho_\Delta = \frac{\mathbf{X}(\Delta)\rho}{\text{Tr}[\mathbf{X}(\Delta)\rho]},$$

for any $\Delta \in \mathcal{B}(\mathbf{R}^n)$ with $\text{Pr}[X \in \Delta | \rho] \neq 0$.

Given an operation-valued measure $\mathbf{X} : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{P}(\tau\mathcal{C}(\mathcal{H}))$, any measurement scheme $\mathcal{X} = [\mathcal{K}, \sigma, \hat{U}, \langle \hat{M}_1, \dots, \hat{M}_n \rangle]$ which satisfies Eq. (2.1) is called a *realization* of \mathbf{X} . An operation-valued measure is called *realizable* if it has at least one realization. An importance of complete positivity for operation-valued measures is clear from the following.

Theorem 2.1. *An operation-valued measure $\mathbf{X} : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{P}(\tau\mathcal{C}(\mathcal{H}))$ is realizable if and only if it is a CPOM. In particular, every CPOM $\mathbf{X} : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{P}(\tau\mathcal{C}(\mathcal{H}))$ has a realization $\mathcal{X} = [\mathcal{K}, \sigma, \hat{U}, \langle \hat{M}_1, \dots, \hat{M}_n \rangle]$ such that σ is a pure state and $\dim(\mathcal{H}) = \dim(\mathcal{K})$.*

For a proof, see [16, Section 5]. A consequence from the above theorem is the following version of the Naimark extension of POM's.

Corollary 2.2. *For any POM $X : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{L}(\mathcal{H})$, there exists a measurement scheme $\mathcal{X} = [\mathcal{K}, |\phi\rangle\langle\phi|, \hat{U}, \langle \hat{M}_1, \dots, \hat{M}_n \rangle]$ satisfying the relation*

$$X(\Delta_1 \times \dots \times \Delta_n) = V^\dagger [\hat{U}^\dagger (1 \otimes M_1(\Delta_1) \cdots M_n(\Delta_n)) \hat{U}] V, \quad (2.2)$$

for all $\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R}^n)$, where V is the isometry from \mathcal{H} to $\mathcal{H} \otimes \mathcal{K}$ such that $V\psi = \psi \otimes \phi$ for all $\psi \in \mathcal{H}$.

A measurement scheme \mathcal{X} satisfying Eq. (2.2) is called an *interacting realization* of a POM X . A definition of the non-interacting version of realizations of POM's appears in [12, p. 68] and it should be noted that an interacting realization determines the state reduction but a non-interacting one does not.

The outcome variable X of a measurement scheme \mathcal{X} is generally called a *quantum random variable* (q.r.v.). Thus any q.r.v. X has a CPOM \mathbf{X} which determines the probability distributions of X . Let $\langle X_1, \dots, X_n \rangle$ be a finite sequence of q.r.v.s of measurement schemes $\mathcal{X}_1, \dots, \mathcal{X}_n$ and $\langle \mathbf{X}_1, \dots, \mathbf{X}_n \rangle$ the corresponding sequence of CPOM's. Then, from (DL1) and (DL2), the joint probability distribution $\Pr[X_1 \in \Delta_1, \dots, X_n \in \Delta_n \|\rho]$, $(\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R}^n))$, of $\langle X_1, \dots, X_n \rangle$ in a state ρ is given by the following *Davies-Lewis formula* [8]:

$$\Pr[X_1 \in \Delta_1, \dots, X_n \in \Delta_n \|\rho] = \text{Tr}[\mathbf{X}_n(\Delta_n) \cdots \mathbf{X}_1(\Delta_1)\rho]. \quad (2.3)$$

Let \hat{H} be the Hamiltonian of the system \mathbf{S} and $\hat{U}_t = e^{-it\hat{H}/\hbar}$ the unitary operator of the time evolution. For the Heisenberg system state ρ , we shall write,

$$\rho(t) = \alpha(t)\rho = \hat{U}_t \rho \hat{U}_t^\dagger,$$

for the time evolution of the states in the Schrödinger picture. Suppose that a finite sequence of measurements corresponding to a sequence $\langle \mathbf{X}_1, \dots, \mathbf{X}_n \rangle$ of CPOM is made successively at time $(0 <) t_1 < \dots < t_n$, where it is supposed that $\tilde{\tau}_i \ll t_{i+1} - t_i$ for the durations $\tilde{\tau}_i$ of the coupling of measurement \mathbf{X}_i . We shall denote by $X_i(t_i)$ the outcome variable of the measurement \mathbf{X}_i at time t_i . Then the joint probability distribution of the sequence $\langle X_1(t_1), \dots, X_n(t_n) \rangle$ of the outcomes in the state $\rho = \rho(0)$ is given by the following *Wigner-Davies-Lewis formula* [26, 8]:

$$\begin{aligned} & \Pr[X_1(t_1) \in \Delta_1, X_2(t_2) \in \Delta_2, \dots, X_n(t_n) \in \Delta_n \|\rho] \\ &= \text{Tr}[\mathbf{X}_n(\Delta_n) \alpha(t_n - t_{n-1}) \cdots \mathbf{X}_2(\Delta_2) \alpha(t_2 - t_1) \mathbf{X}_1(\Delta_1) \alpha(t_1) \rho], \\ & \quad (\Delta_1, \Delta_2, \dots, \Delta_n \in \mathcal{B}(\mathbf{R})). \end{aligned}$$

Let $\mathbf{X} : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{P}(\tau\mathcal{C}(\mathcal{H}))$ be a CPOM and ρ a density operator on \mathcal{H} . A family $\{\rho_x \mid x \in \mathbf{R}^n\}$ of density operators on \mathcal{H} is called a family of *posterior states* for a *prior state* ρ and a CPOM \mathbf{X} if it satisfies the following conditions (PS1)–(PS2):

(PS1) The function $x \mapsto \rho_x$ is strongly Borel measurable.

(PS2) For any $\Delta \in \mathcal{B}(\mathbf{R}^n)$,

$$\int_{\Delta} \rho_x \text{Tr}[\mathbf{X}(dx)\rho] = \mathbf{X}(\Delta)\rho.$$

A family of posterior states always exists for any prior state ρ and it is unique in the following sense: If $\{\rho'_x \mid x \in \mathbf{R}^n\}$ is another family of posterior states for the prior state ρ , then $\rho'_x = \rho_x$ for $\text{Tr}[\mathbf{X}(dx)\rho]$ -almost everywhere [17]. Suppose that a measurement corresponding to \mathbf{X} is made for the system \mathbf{S} in a state ρ at the time of measurement and that the measurement gives the outcome $X = x$ ($x \in \mathbf{R}^n$). Let $\{\rho_x \mid x \in \mathbf{R}^n\}$ be a family of posterior states for the prior state ρ . Then, with probability 1, ρ_x is the state of the system \mathbf{S} at the time just after the measurement.

3. Noise of approximate measurement

Let \mathcal{H} be a Hilbert space corresponding to a quantum system \mathbf{S} . Let A be an observable of \mathbf{S} . In this section, we consider a measuring instrument designed to measure the value of an observable A and discuss the noise contained in outcomes from the measuring instrument. Let X be a q.r.v. representing the outcome from the measuring instrument. Then the probability distribution $\Pr[X \in \Delta | \rho]$ of X in a state ρ of the system \mathbf{S} at the time of measurement is represented by a POM $\hat{\mathbf{X}}$ for some CPOM \mathbf{X} satisfying

$$\Pr[X \in \Delta | \rho] = \text{Tr}[\hat{\mathbf{X}}(\Delta)\rho], \quad (\Delta \in \mathcal{B}(\mathbf{R})).$$

For simplicity of notation, we shall write $X(\Delta) = \hat{\mathbf{X}}(\Delta)$, $(\Delta \in \mathcal{B}(\mathbf{R}))$. Let f be a real Borel function on \mathbf{R} . The *expectation* $\text{Ex}[f(X) | \rho]$ of the q.r.v. $f(X)$ in a state ρ is defined by

$$\text{Ex}[f(X) | \rho] = \int_{\mathbf{R}} f(x) \Pr[X \in dx | \rho],$$

provided the integral is convergent. Denote by $f(\widehat{X})$ the symmetric operator defined by

$$\begin{aligned} \langle \xi | f(\widehat{X}) | \xi \rangle &= \int_{\mathbf{R}} f(x) \langle \xi | X(dx) | \xi \rangle, \quad (\xi \in \text{dom}(f(\widehat{X}))), \\ \text{dom}(f(\widehat{X})) &= \{ \xi \in \mathcal{H} \mid \int_{\mathbf{R}} f(x)^2 \langle \xi | X(dx) | \xi \rangle < \infty \}. \end{aligned} \quad (3.1)$$

Then we have

$$\text{Ex}[f(X) | \psi] = \langle \psi | f(\widehat{X}) | \psi \rangle,$$

for any vector state $\psi \in \text{dom}(f(\widehat{X}))$. The *variance* $\text{Var}[X | \rho]$ and the *standard deviation* $\Delta X[\rho]$ of X in a state ρ are defined in the usual way, i.e.,

$$\begin{aligned} \text{Var}[X | \rho] &= \text{Ex}[X^2 | \rho] - \text{Ex}[X | \rho]^2, \\ \Delta X[\rho] &= \text{Var}[X | \rho]^{1/2}. \end{aligned} \quad (3.2)$$

We say that a POM X has *finite second moment* in a state ρ if $\text{Ex}[X^2 | \rho] < \infty$, or equivalently, if $\Delta X[\rho] < \infty$. Let X be a POM with finite second moment in ρ . Then we have $\text{dom}(\hat{X}) \supset \text{ran}(\sqrt{\rho})$ and that $\hat{X}\sqrt{\rho}$ is a Hilbert-Schmidt operator [12]. Thus, we shall write $\text{Tr}[\hat{X}^2 \rho] = \text{Tr}[(\hat{X}\sqrt{\rho})^\dagger \hat{X}\sqrt{\rho}]$. For any POM's X, Y with $\Delta X[\rho], \Delta Y[\rho] < \infty$, the expression $\text{Tr}[\hat{X}, \hat{Y}] \rho$ is defined by

$$\text{Tr}[\hat{X}, \hat{Y}] \rho = \text{Tr}[(\hat{X}\sqrt{\rho})^\dagger \hat{Y}\sqrt{\rho} - (\hat{Y}\sqrt{\rho})^\dagger \hat{X}\sqrt{\rho}].$$

By the Robertson uncertainty relation [24], for any state ρ and any pair of observables A, B with $\Delta A[\rho], \Delta B[\rho] < \infty$, we have

$$\Delta A[\rho] \Delta B[\rho] \geq \frac{1}{2} |\text{Tr}[\hat{A}, \hat{B}] \rho|. \quad (3.3)$$

The above relation is extended to any pair of POM's by Holevo [12, p.90], i.e., for any pair of POM's X, Y with $\Delta X[\rho], \Delta Y[\rho] < \infty$, we have

$$\Delta X[\rho] \Delta Y[\rho] \geq \frac{1}{2} |\text{Tr}[\hat{X}, \hat{Y}] \rho|. \quad (3.4)$$

We say that a POM $X : \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{L}(\mathcal{H})$ is *compatible* with an observable A (or *A-compatible*, in short) if it satisfies the relation

$$[X(\Delta_1), A(\Delta_2)] = 0,$$

for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$. Let ρ be a state at the time of measurement. For an A -compatible POM X , the joint probability distribution $\Pr[X \in \Delta_1, A \in \Delta_2 | \rho]$ of X and A in a state ρ is given by

$$\Pr[X \in \Delta_1, A \in \Delta_2 | \rho] = \text{Tr}[X(\Delta_1)A(\Delta_2)\rho], \quad (\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})).$$

By a notational convention, for a Borel measure μ on $\mathcal{B}(\mathcal{R}^2)$, we shall write

$$\iint_{\mathbf{R}^2} f(x, y) \nu(dx, dy) = \int_{\mathbf{R}^2} f(x, y) \mu(d(x, y)),$$

where ν is the joint measure on $\mathcal{B}(\mathbf{R}) \times \mathcal{B}(\mathbf{R})$ defined by $\nu(\Delta_1, \Delta_2) = \mu(\Delta_1 \times \Delta_2)$, $(\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R}))$, and we shall write $L^p(\mathbf{R}^2, \nu(dx, dy)) = L^p(\mathbf{R}^2, \mu)$ for the L^p -space of μ . We define the *root-mean-square error* (or *precision*, in short) $\epsilon[X|A, \rho]$ of X for measurement of an observable A in a state ρ by the relation

$$\epsilon[X|A, \rho]^2 = \iint_{\mathbf{R}^2} (x - a)^2 \text{Tr}[X(dx)A(da)\rho]. \quad (3.5)$$

Obviously, $\epsilon[X|A, \rho]$ represents the root-mean-square deviation of the outcome X of the measurement from the outcome A of the ideal measurement, when these two were made simultaneously in the state ρ .

Lemma 3.1. *Let μ be a finite Borel measure on \mathbf{R}^2 . Then the relation*

$$\iint_{\mathbf{R}^2} (x - y)^2 \mu(dx \times dy) = 0 \quad (3.6)$$

holds if and only if for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$,

$$\mu(\Delta_1 \times \Delta_2) = \mu((\Delta_1 \cap \Delta_2) \times \mathbf{R}). \quad (3.7)$$

Proof. Suppose that Eq. (3.6) holds. Let $D = \{(x, y) \in \mathbf{R}^2 | x = y\}$. Then it follows from Eq. (3.6) that $\mu(\mathbf{R}^2 \setminus D) = 0$. Then we have

$$\begin{aligned} \mu(\Delta_1 \times \Delta_2) &= \mu((\Delta_1 \times \Delta_2) \cap D) = \mu((\Delta_1 \cap \Delta_2) \times \mathbf{R}) \cap D \\ &= \mu((\Delta_1 \cap \Delta_2) \times \mathbf{R}). \end{aligned}$$

Conversely suppose that Eq. (3.7) holds. Then we have

$$\begin{aligned} \mu(\Delta_1 \times \Delta_2) &= \mu((\Delta_1 \cap \Delta_2) \times \mathbf{R}) \\ &= \int_{\Delta_1 \cap \Delta_2} \mu(dx \times \mathbf{R}) \\ &= \int_{\Delta_1} \delta_x(\Delta_2) \mu(dx \times \mathbf{R}) \\ &= \int_{\mathbf{R}} \mu(dx \times \mathbf{R}) \int_{\mathbf{R}} \chi_{\Delta_1 \times \Delta_2}(x, y) \delta_x(dy), \end{aligned}$$

where δ_x is the Dirac measure of $x \in \mathbf{R}$. Thus we obtain

$$\begin{aligned} \iint_{\mathbf{R}^2} (x - y)^2 \mu(dx \times dy) &= \int_{\mathbf{R}} \mu(dx \times \mathbf{R}) \int_{\mathbf{R}} (x - y)^2 \delta_x(dy) \\ &= 0. \end{aligned}$$

QED

Theorem 3.2. *An A -compatible POM X satisfies the relation*

$$\epsilon[X\|A, \rho] = 0, \quad (3.8)$$

for all density operator ρ on \mathcal{H} if and only if $X = A$.

Proof. Obviously, if $X = A$ then Eq. (3.8) holds. Suppose that Eq. (3.8) holds. Let $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\mathbf{R})$. Let μ be the Borel measure on \mathbf{R}^2 such that $\mu(\Delta_1 \times \Delta_2) = \text{Tr}[X(\Delta_1)A(\Delta_2)\rho]$, $(\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R}))$. Then since $\epsilon[X\|A, \rho] = 0$, we have $\mu(\Delta \times \mathbf{R}) = \mu(\mathbf{R} \times \Delta)$ from Lemma 3.1, and hence

$$\text{Tr}[A(\Delta)\rho] = \text{Tr}[A(\Delta)X(\mathbf{R})\rho] = \text{Tr}[A(\mathbf{R})X(\Delta)\rho] = \text{Tr}[X(\Delta)\rho].$$

Since ρ and Δ are arbitrary, it is concluded that $A = X$. *QED*

Lemma 3.3. *For any A -compatible POM X , there exists a Hilbert space $\tilde{\mathcal{H}}$, an isometry $V : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ and self-adjoint operators \tilde{X} and \tilde{A} on $\tilde{\mathcal{H}}$ satisfying the following conditions:*

- (1) $[\tilde{X}, \tilde{A}] = 0$ and $\tilde{A}V = V\hat{A}$.
- (2) For any $\rho \in \mathcal{S}(\mathcal{H})$ and $f \in L^1(\mathbf{R}^2, \text{Tr}[X(dx)A(da)\rho])$,

$$\iint_{\mathbf{R}^2} f(x, a) \text{Tr}[X(dx)A(da)\rho] = \text{Tr}[f(\tilde{X}, \tilde{A})V\rho V^\dagger].$$

Proof. Let $M : \mathcal{B}(\mathbf{R}^2) \rightarrow \mathcal{L}(\mathcal{H})$ be a POM such that $M(\Delta_1 \times \Delta_2) = X(\Delta_1)A(\Delta_2)$, for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$. Then, by the Naimark extension of M , there exist a Hilbert space $\tilde{\mathcal{H}}$, an isometry $V : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ and a projection valued measure $E : \mathcal{B}(\mathbf{R}^2) \rightarrow \mathcal{L}(\tilde{\mathcal{H}})$ such that $M(\Delta_1 \times \Delta_2) = V^\dagger E(\Delta_1 \times \Delta_2) V$ for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$. Let \tilde{X} and \tilde{A} be self-adjoint operators on $\tilde{\mathcal{H}}$ defined by

$$\begin{aligned} \tilde{X} &= \int_{\mathbf{R}} x E(dx \times \mathbf{R}), \\ \tilde{A} &= \int_{\mathbf{R}} a E(\mathbf{R} \times da). \end{aligned}$$

Then the assertion follows from a straightforward verification. *QED*

An A -compatible POM X is said to be *unbiased* if $\hat{A} = \hat{X}$; in this case, we have. $\text{Ex}[A\|\rho] = \text{Ex}[X\|\rho]$ for all state ρ with $\Delta X[\rho] < \infty$.

Theorem 3.4. *Let X be an unbiased A -compatible POM. Then for any state ρ with $\Delta X[\rho] < \infty$, we have*

$$\epsilon[X\|A, \rho]^2 = \Delta X[\rho]^2 - \Delta A[\rho]^2. \quad (3.9)$$

Proof. By Lemma 3.3, we have

$$\begin{aligned}
\epsilon[X|A, \rho]^2 &= \iint_{\mathbf{R}^2} (x - a)^2 \text{Tr}[X(dx)A(da)\rho] \\
&= \text{Tr}[(\tilde{X} - \tilde{A})^2 V \rho V^\dagger] \\
&= \text{Tr}[\tilde{X}^2 V \rho V^\dagger] - \text{Tr}[\tilde{A}^2 V \rho V^\dagger] \\
&= \Delta X[\rho]^2 - \Delta A[\rho]^2.
\end{aligned}$$

QED

Remark. When X is A -compatible but $\hat{A} \neq \hat{X}$, we have

$$\epsilon[X|A, \rho]^2 = \Delta X[\rho]^2 - \Delta \hat{X}[\rho]^2 + \text{Tr}[(\hat{A} - \hat{X})^2 \rho],$$

where $\Delta \hat{X}[\rho]^2 = \text{Tr}[\hat{X}^2 \rho] - \text{Tr}[\hat{X} \rho]^2$. It follows that $\epsilon[X|A, \rho]$ has a lower bound such that $\epsilon[X|A, \rho]^2 \geq \text{Tr}[(\hat{A} - \hat{X})^2 \rho]$.

4. Uncertainty principle for joint measurements

Consider a measuring instrument with two output variables X, Y designed to measure the values of observables A, B of a quantum system \mathbf{S} described by a Hilbert space \mathcal{H} . Let $M : \mathcal{B}(\mathbf{R}^2) \rightarrow \mathcal{L}(\mathcal{H})$ be the joint POM of the pair $\langle X, Y \rangle$, and ρ be a state of \mathbf{S} at the time of measurement. Then we have

$$\text{Pr}[X \in \Delta_1, Y \in \Delta_2 | \rho] = \text{Tr}[M(\Delta_1 \times \Delta_2)\rho],$$

for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$. Let $\langle M_X, M_Y \rangle$ be the pair of marginal POM's of M , i.e., $M_X(\Delta) = M(\Delta \times \mathbf{R})$, $M_Y(\Delta) = M(\mathbf{R} \times \Delta)$, ($\Delta \in \mathcal{B}(\mathbf{R})$). Then M_X and M_Y are the POM's of q.r.v.s X and Y , respectively, and hence it is natural to assume that M_X is an unbiased A -compatible POM and that M_Y is an unbiased B -compatible POM. In this case, it is known [2, 11, 27, 1] that $\Delta X[\rho]$ and $\Delta Y[\rho]$ obeys a more stringent uncertainty relation than the Robertson-Holevo relation (3.4). A general proof of this fact is given below along with the ideas in [1].

A pair $\langle X, Y \rangle$ of POM's is called a *coexistent* pair if there is a POM $M : \mathcal{B}(\mathbf{R}^2) \rightarrow \mathcal{L}(\mathcal{H})$ such that $X(\Delta) = M(\Delta \times \mathbf{R})$, $Y(\Delta) = M(\mathbf{R} \times \Delta)$, for all $\Delta \in \mathcal{B}(\mathbf{R})$.

Theorem 4.1. *Let \hat{A}, \hat{B} be self-adjoint operators on a Hilbert space \mathcal{H} . Let $\langle X, Y \rangle$ be a coexistent pair of POM's such that X is an unbiased A -compatible POM and Y is an unbiased B -compatible POM. Then, for any state ρ with $\Delta X[\rho], \Delta Y[\rho] < \infty$, we have*

$$(1) \epsilon[X|A, \rho] \epsilon[Y|B, \rho] \geq \frac{1}{2} |\text{Tr}[\hat{A}, \hat{B} \rho]|,$$

$$(2) \Delta X[\rho] \Delta Y[\rho] \geq |\text{Tr}[\hat{A}, \hat{B} \rho]|.$$

Proof. For simplicity, we shall prove the case where $\rho = |\psi\rangle\langle\psi|$. Let $M : \mathcal{B}(\mathbf{R}^2) \rightarrow \mathcal{L}(\mathcal{H})$ be a POM such that $M(\Delta \times \mathbf{R}) = X(\Delta)$ and $M(\mathbf{R} \times \Delta) = Y(\Delta)$. Let $\mathcal{X} =$

$[\mathcal{K}, |\phi\rangle\langle\phi|, \hat{U}, \langle\hat{M}_1, \hat{M}_2\rangle]$ be an interacting realization of M . Set $X_1 = X$, $X_2 = Y$, $A_1 = A$, and $A_2 = B$. Define noise operators \hat{N}_i ($i = 1, 2$) by the relation

$$\hat{N}_i = \hat{U}^\dagger(1 \otimes \hat{M}_i)\hat{U} - \hat{A}_i \otimes 1.$$

Then we have

$$\begin{aligned} \langle\psi \otimes \phi | \hat{N}_i | \psi \otimes \phi\rangle &= \langle\psi \otimes \phi | \hat{U}^\dagger(1 \otimes \hat{M}_i)\hat{U} | \psi \otimes \phi\rangle - \langle\psi | \hat{A}_i | \psi\rangle \\ &= \langle\psi | \hat{X}_i | \psi\rangle - \langle\psi | \hat{A}_i | \psi\rangle \\ &= 0, \end{aligned}$$

and hence

$$\begin{aligned} \Delta\hat{N}_i[\psi \otimes \phi]^2 &= \langle\psi \otimes \phi | \hat{N}_i^2 | \psi \otimes \phi\rangle \\ &= \langle\psi \otimes \phi | (\hat{U}^\dagger(1 \otimes \hat{M}_i)\hat{U} - \hat{A}_i \otimes 1)^2 | \psi \otimes \phi\rangle \\ &= \langle\psi | \hat{X}_i^2 | \psi\rangle - \langle\psi | \hat{A}_i^2 | \psi\rangle \\ &= \epsilon[X_i | A_i, \psi]^2. \end{aligned}$$

On the other hand, from the relations ($i, j = 1, 2$)

$$\begin{aligned} \langle\psi \otimes \phi | \hat{U}^\dagger(1 \otimes \hat{M}_i)\hat{U}(\hat{A}_j \otimes 1) | \psi \otimes \phi\rangle &= \langle\psi \otimes \phi | \hat{U}^\dagger(1 \otimes \hat{M}_i)\hat{U} | (\hat{A}_j \psi) \otimes \phi\rangle \\ &= \langle\psi | \hat{A}_i \hat{A}_j | \psi\rangle, \end{aligned}$$

we have

$$\langle\psi \otimes \phi | [\hat{N}_1, \hat{N}_2] | \psi \otimes \phi\rangle = \langle\psi | [\hat{A}, \hat{B}] | \psi\rangle.$$

Thus by the Robertson uncertainty relation we have

$$\begin{aligned} \epsilon[X | A, \psi] \epsilon[Y | B, \psi] &= \Delta\hat{N}_1[\psi \otimes \phi] \Delta\hat{N}_2[\psi \otimes \phi] \\ &\geq \frac{1}{2} |\langle\psi \otimes \phi | [\hat{N}_1, \hat{N}_2] | \psi \otimes \phi\rangle| \\ &= \frac{1}{2} |\langle\psi | [\hat{A}, \hat{B}] | \psi\rangle|. \end{aligned}$$

This concludes (1). From this relation, Theorem 3.4 and the Robertson uncertainty relation, we obtain

$$\begin{aligned} \Delta X[\rho]^2 \Delta Y[\rho]^2 &= (\epsilon[X | A, \rho]^2 + \Delta A[\rho]^2)(\epsilon[Y | B, \rho]^2 + \Delta B[\rho]^2) \\ &\geq (\epsilon[X | A, \rho] \epsilon[Y | B, \rho] + \Delta A[\rho] \Delta B[\rho])^2 \\ &\geq |\text{Tr}[\hat{A}, \hat{B}] \rho|^2. \end{aligned}$$

This proves (2). *QED*

5. Standard quantum limit for repeated measurements

Let $\mathbf{X} : \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{P}(\tau c(\mathcal{H}))$ be a CPOM and A an observable of a system \mathbf{S} corresponding to \mathcal{H} . We define the *root-mean-square scatter* (or *resolution*, in short) $\sigma[\mathbf{X}||A, \rho]$ of a CPOM \mathbf{X} for measurement of an observable A in a state ρ by the relation

$$\sigma[\mathbf{X}||A, \rho]^2 = \iint_{\mathbf{R}^2} (x - a)^2 \text{Tr}[A(da)\mathbf{X}(dx)\rho]. \quad (5.1)$$

Let $\{\rho_x \mid x \in \mathbf{R}\}$ be a family of posterior state for a prior state ρ and \mathbf{X} . Then, we have

$$\sigma[\mathbf{X}||A, \rho]^2 = \int_{\mathbf{R}} \text{Tr}[\mathbf{X}(dx)\rho] \int_{\mathbf{R}} (x - a)^2 \text{Tr}[A(da)\rho_x]. \quad (5.2)$$

Theorem 5.1. *Let \hat{A} be a self-adjoint operator on a Hilbert space \mathcal{H} . Let $\mathbf{X} : \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{P}(\tau c(\mathcal{H}))$ be a CPOM, ρ a density operator on \mathcal{H} with $\Delta A[\mathbf{X}(\mathbf{R})\rho]$, $\Delta \hat{\mathbf{X}}[\rho] < \infty$ and $\{\rho_x \mid x \in \mathbf{R}\}$ a family of posterior states for ρ and \mathbf{X} . Then we have*

$$\sigma[\mathbf{X}||A, \rho]^2 = \int_{\mathbf{R}} \Delta A[\rho_x]^2 \text{Tr}[\mathbf{X}(dx)\rho] + \int_{\mathbf{R}} (\text{Tr}[\hat{A}\rho_x] - x)^2 \text{Tr}[\mathbf{X}(dx)\rho].$$

Proof. From $\Delta A[\mathbf{X}(\mathbf{R})\rho] < \infty$, we obtain

$$\int_{\mathbf{R}} \text{Tr}[\hat{A}^2 \rho_x] \text{Tr}[\mathbf{X}(dx)\rho] = \text{Tr}[\hat{A}^2 \mathbf{X}(\mathbf{R})\rho] < \infty,$$

and hence $\text{Tr}[\hat{A}^2 \rho_x] < \infty$, $\text{Tr}[\mathbf{X}(dx)\rho]$ -almost everywhere. Thus the assertion follows from Eq. (5.2) and the relations

$$\begin{aligned} \int_{\mathbf{R}} (a - x)^2 \text{Tr}[A(da)\rho_x] &= \text{Tr}[\hat{A}^2 \rho_x] - 2x \text{Tr}[\hat{A}\rho_x] + x^2 \\ &= \text{Tr}[\hat{A}^2 \rho_x] - \text{Tr}[\hat{A}\rho_x]^2 + (\text{Tr}[\hat{A}\rho_x] - x)^2 \\ &= \Delta A[\rho_x]^2 + (\text{Tr}[\hat{A}\rho_x] - x)^2. \end{aligned}$$

QED

Let \mathbf{X} be a CPOM of a measuring instrument with one output variable X designed to make an unbiased measurement of an observable A of a system \mathbf{S} corresponding to a Hilbert space \mathcal{H} . Suppose that the system \mathbf{S} undergoes unitary evolution during the time τ between two identical measurements described by the CPOM \mathbf{X} . Let \hat{U}_τ be the unitary operator of the time evolution of the system \mathbf{S} , i.e., $\hat{U}_\tau = e^{-i\tau \hat{H}/\hbar}$, where \hat{H} is the Hamiltonian of \mathbf{S} . Suppose that the system \mathbf{S} is in a state ρ at the time of the first measurement. Then at the time just after the first measurement (say, $t = 0$) the system is in a posterior state ρ_x with the probability distribution $\text{Pr}[X \in dx|\rho] = \text{Tr}[\mathbf{X}(dx)\rho]$. From this outcome $X = x$, the observer makes a prediction $X(\tau) = h(x)$ for the outcome of the second measurement at $t = \tau$. Then the squared uncertainty of this prediction is

$$\begin{aligned} \Delta[\tau, \rho, x]^2 &= \int_{\mathbf{R}} (a - h(x))^2 \text{Pr}[X \in da|\rho_x(\tau)] \\ &= \int_{\mathbf{R}} (a - h(x))^2 \text{Tr}[\mathbf{X}(da)\alpha(\tau)\rho_x]. \end{aligned} \quad (5.3)$$

As to determination of $h(x)$, the following mean-value-prediction strategy is naturally adopted:

$$h(x) = \text{Tr}[\rho_x \hat{A}(\tau)], \quad (5.4)$$

where

$$\hat{A}(\tau) = \hat{U}_\tau^\dagger \hat{A}(0) \hat{U}_\tau. \quad (5.5)$$

The predictive uncertainty $\Delta[\tau, \rho]$ of this repeated measurement with the prior state ρ and the time duration τ is defined as the root-mean square of $\Delta[\tau, \rho, x]$ over all outcomes $X = x$ of the first measurement, i.e.,

$$\begin{aligned} \Delta[\tau, \rho]^2 &= \int \Delta[\tau, \rho, x]^2 \text{Pr}[X \in dx | \rho] \\ &= \int \int_{\mathbf{R}^2} (a - h(x))^2 \text{Tr}[\mathbf{X}(da) \alpha(\tau) \mathbf{X}(dx) \rho]. \end{aligned} \quad (5.6)$$

Theorem 5.2. *Let $\mathbf{X} : \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{P}(\tau c(\mathcal{H}))$ be an unbiased A -compatible CPOM and ρ a density operator on \mathcal{H} with $\Delta A(0)[\mathbf{X}(\mathbf{R})\rho]$, $\Delta A(\tau)[\mathbf{X}(\mathbf{R})\rho] < \infty$. If the relation*

$$\sigma[\mathbf{X} \| A, \rho] \leq \epsilon[\hat{\mathbf{X}} \| A, \alpha(\tau) \mathbf{X}(\mathbf{R})\rho], \quad (5.7)$$

holds then we have

$$\Delta[\tau, \rho]^2 \geq \left| \text{Tr}[\hat{A}(0), \hat{A}(\tau)] \mathbf{X}(\mathbf{R})\rho \right|. \quad (5.8)$$

Proof. From Eqs. (5.3)–(5.5) and Theorem 3.4,

$$\begin{aligned} \Delta[\tau, \rho, x]^2 &= \Delta X[\alpha(\tau) \rho_x]^2 \\ &= \epsilon[X \| \alpha(\tau) \rho_x]^2 + \Delta A(\tau)[\rho_x]^2, \end{aligned}$$

and hence by Theorem 5.1 and the Robertson uncertainty principle,

$$\begin{aligned} \Delta[\tau, \rho]^2 &= \int_{\mathbf{R}} \epsilon[\mathbf{X} \| \alpha(\tau) \rho_x]^2 + \Delta A(\tau)[\rho_x]^2 \text{Tr}[\mathbf{X}(dx) \rho] \\ &= \epsilon[\hat{\mathbf{X}} \| A, \alpha(\tau) \mathbf{X}(\mathbf{R})\rho]^2 + \int_{\mathbf{R}} \Delta A(\tau)[\rho_x]^2 \text{Tr}[\mathbf{X}(dx) \rho] \\ &\geq \sigma[\mathbf{X} \| A, \rho]^2 + \int_{\mathbf{R}} \Delta A(\tau)[\rho_x]^2 \text{Tr}[\mathbf{X}(dx) \rho] \\ &\geq \int_{\mathbf{R}} \Delta A(0)[\rho_x]^2 + \Delta A(\tau)[\rho_x]^2 \text{Tr}[\mathbf{X}(dx) \rho] \\ &\geq \int_{\mathbf{R}} 2\Delta A(0)[\rho_x] \Delta A(\tau)[\rho_x] \text{Tr}[\mathbf{X}(dx) \rho] \\ &\geq \int_{\mathbf{R}} \left| \text{Tr}[\hat{A}(0), \hat{A}(\tau)] \rho_x \right| \text{Tr}[\mathbf{X}(dx) \rho] \\ &\geq \left| \text{Tr}[\hat{A}(0), \hat{A}(\tau)] \mathbf{X}(\mathbf{R})\rho \right|. \end{aligned}$$

QED

The bound (5.8) is called the *standard quantum limit* (SQL) for repeated measurements with interval τ of an observable A . For the case where A is the position observable x of a free-mass m , relation (5.8) is reduced to the relation

$$\Delta[\tau, \rho]^2 \geq \frac{\hbar\tau}{m}, \quad (5.9)$$

which was posed in [4, 6] and the validity of this standard quantum limit was the subject of a long controversy [28, 5, 14, 21]. By the above theorem, any measuring instrument which beats the SQL must have the resolution larger than the precision. In [28], Yuen pointed out a flaw in the original derivation of the SQL (5.9) and proposed an idea of using contractive states to beat the SQL. A model which clears the above condition and beats the SQL was successfully constructed in our previous work [21, 22] as a realization of Gordon-Louisell measurement $\{|\mu\nu a\omega\rangle\langle a|\}$ [9], where $|\mu\nu a\omega\rangle$ is a contractive state and $|a\rangle$ is a position eigenstate. Many linear-coupling models of position measurements which violates condition (5.7) are constructed in [23].

6. Concluding remarks

We have discussed quantum mechanical limitations on joint measurements and repeated measurements of a single object. It is shown that the uncertainty principle for joint measurements of noncommuting observables holds generally with a more stringent limit than the one usually supposed by the Robertson uncertainty relation. On the other hand, the SQL, which is also usually supposed from the Robertson uncertain relation, for repeated measurements of a single observable does not generally hold unless a certain additional condition is satisfied. The difference between these two problem is clear from the difference between those two uncertainties defined by Eq. (3.5) and Eq. (5.6) for which the optimizations are required. The crucial point is that in the latter problem we can use the result of the first measurement to predict the result of the second and hence the prediction can be based on posterior probability. However, in the problem of joint measurements, we are required to predict two outcomes only from prior probability given by the prior state. Thus we can circumvent the uncertainty principle in the problem of repeated measurements, when the measurement changes the prior state to the posterior state which has deterministic information about the *future value* of the observable to be measured. Of course, this future value must be significantly uncertain, if the prior state is of deterministic information about the *present value* and the measurement is *not* carried out. However, some measurement can give this present value precisely and further leaves the object in a state with deterministic information about the future value. Thus monitoring a mass in this way can give a precise information about *classical* force which drives the mass.

The author wishes to thank Professor Horace P. Yuen for hospitality at Northwestern University and Professor Roy J. Glauber for hospitality at Harvard University during his leave in 1988–1990. This work is supported in part by Hamamatsu Photonics K. K.

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